In 2-D

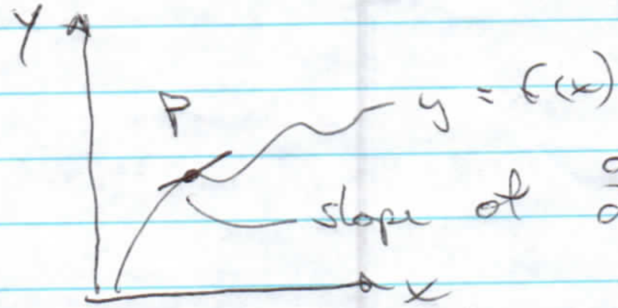
Vector with slope m is

$$\underline{A} = \hat{i} + m\hat{j} \quad \text{or} \quad \frac{1}{m}\hat{i} + \hat{j}$$

- Orthogonal $\underline{A}_\perp = m\hat{i} - \hat{j}$ or $-m\hat{i} + \hat{j}$

- Vector along curve $y = f(x)$

$$\underline{A} = t\hat{i} + f(t)\hat{j}$$



slope of $\frac{dA}{dt}$ at $P = f'(P)$

- Build line from pt + dir
- Build plane from 3 pts, pt + dir
- Calc I stuff

Converting from $\underline{r}(t)$ to $\underline{r}(s)$

$$\hat{T} = \frac{\underline{v}}{|\underline{v}|} \text{ when } \underline{v} = \frac{d\underline{r}}{dt}$$

$$\hat{T} = d\underline{r}/ds$$

$$\hat{N} = \frac{\frac{d\hat{T}}{ds}}{\left|\frac{d\hat{T}}{ds}\right|} = \frac{\frac{d\hat{T}}{dt}\cancel{\frac{dt}{ds}}}{\left|\frac{d\hat{T}}{dt}\cancel{\frac{dt}{ds}}\right|}$$

since $\frac{ds}{dt} > 0$

$$\frac{d\hat{T}}{ds} = \kappa \hat{N}$$

$$\hat{B} = \hat{T} \times \hat{N}$$

$$\hat{B} = \hat{T} \times \hat{N}$$

$$\frac{d\hat{B}}{ds} = \frac{d\hat{B}}{dt}\frac{dt}{ds} = -\tau \hat{N}$$

so $\frac{d\hat{B}}{dt} = |\underline{v}|(-\tau)\hat{N}$

$$\frac{d\hat{B}}{ds} = (-\tau)\hat{N}$$

Main idea

$$\frac{d(\;)}{ds} = \frac{d(\;)}{dt}\frac{dt}{ds} \quad \text{where } \frac{ds}{dt} = |\underline{v}|$$

Ex 1

• Do these limits exist?

a) $\lim\limits_{\to(0,0)} = \dfrac{\tan(x+y)}{x+y} = ?$

b) $\lim\limits_{\to(0,0)} \dfrac{x+y}{x-y}$

c) $\lim\limits_{\to(0,0)} \dfrac{x^2}{x^2-y}$

SOL'N.

a) yes

b) No

c) No

Ex) Given $\underline{v} = -\sin t\,\hat{i} + t^3\hat{j} + e^t\hat{k}$

and $\underline{r}(t=0) = 2\hat{i} + \hat{j} + 3\hat{k}$

Calc. $\underline{r}(t)$ and $\underline{a}(t)$

Sol'n. $\underline{a} = \dfrac{d\underline{v}}{dt} = -\cos t\,\hat{i} + 3t^2\hat{j} + e^t\hat{k}$

and $\underline{r} = \int\left(\dfrac{d\underline{r}}{dt}\right)dt = \int\left(-\sin t\,\hat{i} + t^3\hat{j} + e^t\hat{k}\right)dt$

$= \cos t\,\hat{i} + \dfrac{t^4}{4}\hat{j} + e^t\hat{k} + \underline{c}$

but $\underline{r}(0) = \hat{i} + 0\hat{j} + \hat{k} + \underline{c} = 2\hat{i} + \hat{j} + 3\hat{k}$

so $\underline{c} = \hat{i} + \hat{j} + 2\hat{k}$

thus finally $\underline{r}(t) = (\cos t + 1)\hat{i} + \left(\dfrac{t^4}{4} + 1\right)\hat{j} + \left(e^t + 2\right)\hat{k}$

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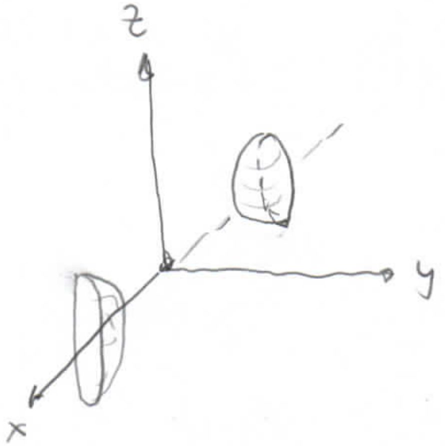
Ex what surfaces are these?

① $x^2 - y^2 - \frac{z^2}{4} = 1$ ② $x^2 - y^2 + z = 1$ ③ $x^2 + y + z^2 = 0$ or $= 2$

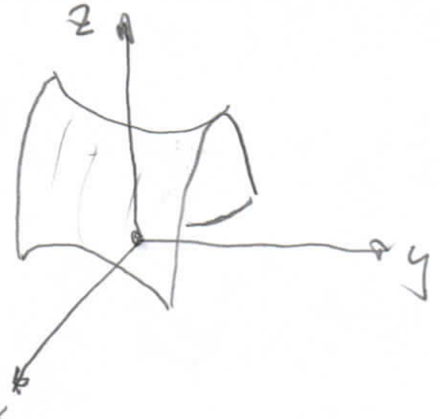
Sol'n.

①



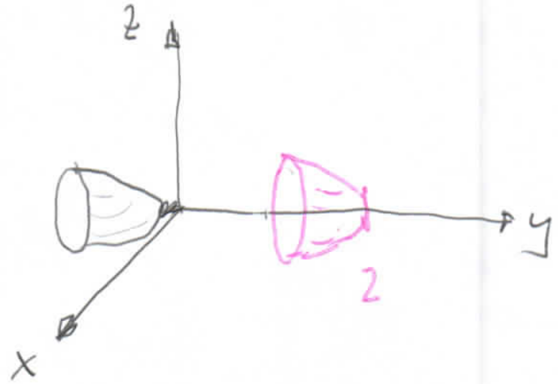
Hyp. of 2-sheets

②



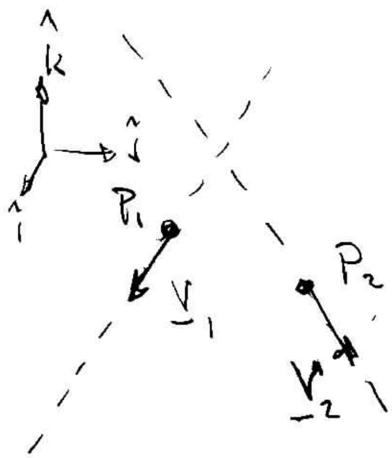
Hyp-parab.

③



Parab.

Ex Consider the 2 lines $\underline{r}_1 = \underline{OP}_1 + \alpha \underline{V}_1$ $\quad \underline{r}_2 = \underline{OP}_2 + \beta \underline{V}_2$



• Find the closest distance between the lines.

Sol'n. • Form $\underline{n} = \underline{V}_1 \times \underline{V}_2$

• Determine the plane with normal $\underline{n}$ containing P_2. Call it plane$_2$ $\quad ax + by + cz = d$

• Now find distance from P_1 to the plane. (old problem.)

Additional thoughts:

• What 2 points on $\underline{r}_1$ and $\underline{r}_2$ are closest?

Ex1 Given path $\underline{r}(t) = t^3\hat{i} + 2t^2\hat{j} + 4t^3\hat{k}$.

Find arc length $s(T)$.

SOL'N. Calc. $\underline{v} = 3t^2\hat{i} + 4t\hat{j} + 8\hat{j}\hat{k}$

so $|\underline{v}| = \sqrt{9t^4 + 80t^2} = t\sqrt{9t^2+80}$

then $s = \int_{t=0}^{T} |\underline{v}|\,dt = \frac{1}{18}\int_{t=0}^{T} 18t\sqrt{9t^2+80}\,dt$

$= \frac{1}{18}\,\frac{2}{3}\sqrt{9t^2+80}\,\Big|_{t=0}^{T} = \frac{1}{27}\left[\sqrt{9T^2+80} - \sqrt{80}\right]$

Additional thoughts:

- S'pose I wanted s from $t=-1$ to 0? Big trouble! s is negative.
- How to correct the problem?

$s = \int_{t=-1}^{0} |\underline{v}|\,dt = \int_{t=-1}^{0} |t|\sqrt{9t^2+80}\,dt \quad \ldots$

Ex] Given $\underset{\sim}{r}_1(t) = (1+t)\hat{i} + (2+3t)\hat{j} + (3+4t)\hat{k}$

and $\underset{\sim}{r}_2(t) = (3+2s)\hat{i} + (6+4s)\hat{j} + (9+6s)\hat{k}$

Find • the direction of each line

- the intersection point (if any)
- $\measuredangle$ between them

Sol'n • Set $1+t = 3+2s$ and $2+3t = 6+4s$ ... $t=0,\ s=-1$

so coord. is $(1, 2, 3)$.

- Direction of each line is $\underset{\sim}{V}_1 = \hat{i} + 3\hat{j} + 4\hat{k} = \underset{\sim}{D}_1$

$\underset{\sim}{V}_2 = 2\hat{i} + 4\hat{j} + 6\hat{k} = \underset{\sim}{D}_2$

- $\underset{\sim}{D}_1 \cdot \underset{\sim}{D}_2 = |\underset{\sim}{D}_1||\underset{\sim}{D}_2| \cos\theta = 2+12+24 = 38$

so $\cos\theta = 38 / |\underset{\sim}{D}_1||\underset{\sim}{D}_2|$

Additional thoughts:

- Determine the equation of the plane formed by the paths $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$.
- What happens if you use t^2 or $\sqrt{t}$ instead of t in $\underset{\sim}{r}(t)$ & $\underset{\sim}{r}_2(t)$?

• FIND τ for the paths:

① $\underline{r}(t) = t^2\hat{i} + t^3\hat{j} + (9t^2 + 5t^3)\hat{k}$

② $\underline{r}(t) = (5 - 5t - 5\sin^2 t)\hat{i} + t\hat{j} + \cos^2 t\,\hat{k}$

③ $\underline{r}(t) = t\hat{i} + (1-t)\hat{j} + [1 - 2t + (t-1)]\hat{k}$

Sol'n ① path is on plane

$$\underline{r}(t) = t^2\hat{i} + t^3\hat{j} + \underbrace{(9t^2 + 5t^3)}\hat{k}$$

$$z = 9x + 5y \Rightarrow \tau = 0$$

②

$$\underline{r}(t) = \underbrace{(5 - 5t - 5\sin^2 t)}\hat{i} + t\hat{j} + \cos^2 t\,\hat{k}$$

$$x = 5(1 - \sin^2 t) - 5t$$

$$x = 5\cos^2 t - 5t$$

$$x = 5(z - y) \qquad \text{path on plane} \Rightarrow \tau = 0$$

③

$$\underline{r}(t) = t\hat{i} + \underbrace{(1-t)}_{y = 1 - x}\hat{j} + \underbrace{[1 - 2t + (t-1)]}_{z = 1 - 2x - y}\hat{k}$$

path is on intersection of 2 planes so it is a straight line!

$\Rightarrow$ τ cannot be determined

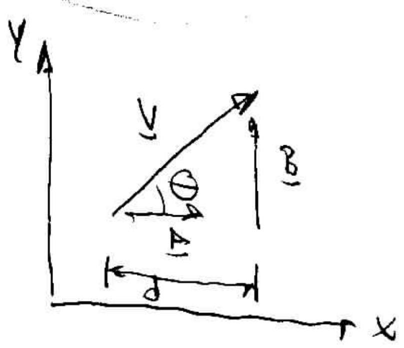
Also note $|\underline{v}| = c$

• other thoughts: can you find any of $\kappa, \tau, \hat{T}, \hat{N}, \hat{B}$?

Ex| Given vectors $\underline{V} = 1\hat{i} + 3\hat{j} + 3\hat{k}$, and $\underline{A} = 2\hat{i} + \hat{j} + 3\hat{k}$

Decomposit into $V = |\underline{V}|\hat{V}$ and $\underline{V} = (\ )\underline{A}_{\parallel} + (\ )\underline{A}_{\perp}$

Sol'n.



$$d = |\underline{V}| \cos\theta$$

but $\underline{A} \circ \underline{V} = |\underline{V}|\,|\underline{A}| \cos\theta$

so

$$d = \frac{\underline{A} \circ \underline{V}}{|\underline{A}|} = |\underline{V}| \cos\theta$$

So

$$\operatorname{proj}_{\underline{A}} \underline{V} = \underbrace{\left(\frac{\underline{A} \circ \underline{V}}{|\underline{A}|}\right)}_{(\text{mag.})} \underbrace{\hat{A}}_{\text{dir}} = \frac{\underline{A} \circ \underline{V}}{|\underline{A}|^2} \underline{A} = \frac{2+3+9}{\sqrt{4+1+9}^{\,2}} \left(2\hat{i} + \hat{j} + 3\hat{k}\right)$$

$$= \sqrt{14} \left(2\hat{i} + \hat{j} + 3\hat{k}\right)$$

$$\underline{B} = \underline{V} - \operatorname{proj}_{\underline{A}} \underline{V} = \left(1\hat{i} + 3\hat{j} + 3\hat{k}\right) - \sqrt{14}\left(2\hat{i} + \hat{j} + 3\hat{k}\right)$$

$$= \left(1 - 2\sqrt{14}\right)\hat{i} + \left(3 - \sqrt{14}\right)\hat{j} + \left(3 - 3\sqrt{14}\right)\hat{k}$$

Finally

$$\underline{V} = \underbrace{\operatorname{proj}_{\underline{A}} \underline{V}}_{\parallel \underline{A}} + \underbrace{\underline{B}}_{\perp \underline{A}}$$

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